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On the Constructive Solution of Nonlinear Functional Equations

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INTRODUCTION

In the recent development of the theory of nonlinear operators in Banach spaces, a number of general existence theorems have been established for various classes of mappings by the use of compactness, convexity, or topological arguments that do not give a constructive procedure for the generation of the solutions thus proved to exist. Even in cases when the solutions are unique and one obtains them as limits of a precisely determined sequence of approximants, they often fail to satisfy an important and useful principle of constructivity in that the procedures have no effective control provided for the error at each state of the approximation. This is particularly the case for the Galerkin approximations used in the theory of operators of monotone type and for various fixed-point methods used in the theory of nonlinear accretive operators.

About a decade ago, the writer developed existence results which satisfied these principles of constructivity for the case of a broad class of continuous monotone mappings in Hilbert spaces, and more generally, for continuous accretive mappings in Banach spaces X whose conjugate spaces X^* are uniformly convex. These results were stated in the rather inaccessible paper [1] and developed in detail in the middle of a lengthy treatment of accretive operators in the writer's paper [2].

It is our object here to develop these results explicitly and in detail in connection with the problem of constructivity. Our renewed interest in this question was stimulated by the recent paper by Bruck [3], who has developed an iteration procedure to obtain the solution of the equation $(I + T)(u) = 0$ for a continuous, bounded monotone operator T on a Hilbert space H with explicit control of the error. Bruck's result has the curious feature that it depends on the assumption of the prior existence of the solution. Although the procedure we give is not an iteration procedure it corresponds to a simpler intuitive picture of the situation in the general context of accretive operators and the error control does not depend upon the assumption of the existence of a solution.

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These results, particularly when applied to the case of monotone operators in Hilbert space, raise very forcefully the problem of whether or not constructive solutions can be found for equations of the form $T(u) = f$ for the case of monotone mappings T for a reflexive Banach space X to its conjugate space X^*, whether by Galerkin or other methods under assumptions upon T of a comparable generality to those presented below.

Section 1

Let X be a Banach space, X^* its conjugate space, (w, u) the pairing between w in X^* and u in X. We assume throughout this section that X^* is uniformly convex and we consider some related results in more general Banach spaces in Section 2.

The (normalized) duality mapping J of X into X^* is given by the conditions

$$(J(u), u) = \| u \|^2, \qquad \| Ju \| = \| u \|.$$

Since X^* is assumed to be uniformly convex, J is uniformly continuous from bounded subsets of X to X^*. Let $R > 0$ be given. Then, in particular, there exists a nonnegative real-valued continuous function w on $B_{2R}(0, X)$ with $w(0) = 0$ such that

$$\| J(u) - J(v)\| \leqslant w(\| u - v \|)$$

for each u and v in $B_{2R}(0, X)$.

Definition 1. If $R > 0$ is given and T is a mapping of $B_R(0, X)$ into X, then T is said to be accretive if for all u and v in $B_R(0, X)$ we have

$$(T(u) - T(v), J(u) - v)) \geqslant 0.$$

Definition 2. The mapping T of $B_R(0, X)$ into X is said to lie in the class (A) if its satisfies the following conditions:

(1) T is an accretive continuous mapping of $B_R(0, X)$ into X.

(2) There exists R_1 with $0 < R_1 < R$ such that for all u in X with $R_1 \leqslant \| u \| \leqslant R$,

$$(T(u), J(u)) \geqslant 0.$$

(3) There exists a constant $M > 0$ such that for all u in $B_R(0, X)$,

$$\| T(u)\| \leqslant M.$$

(4) There exists $\delta > 0$ and a continuous function q from X to the non-negative reals with $q(0) = 0$ such that if u and v are a pair of elements with $| T(u)| \leqslant \delta$, $| T(v)| \leqslant \delta$, then

$$\| u - v \| \leqslant q(\| T(u) - T(v)\|).$$

Note that if T is any accretive mapping and $\xi > 0$, then condition (4) is always satisfied for the mapping $(T + \xi I)$ with $q(r) = \xi^{-1}r$.

For any mapping T which lies in the class (A) and any initial value v_0 lying in $B_{R_1}(0, X)$, we define a sequence of iterates by the following procedure:

DEFINITION 3. Let r and n be two positive integers, v_0 an element of $B_{R_1}(0, X)$. We define a sequence $\{v_k\}$ (depending upon v_0, r, and n) by the prescription

$$v_k = (1 - (n + k)^{-1})\, v_{k-1} - n^{-1}T(v_{k-1}), \qquad 1 \leqslant k \leqslant rn.$$

PROPOSITION 1. *Suppose that r and n are chosen to satisfy the two following inequalities:*

$$n^{-1}(M + R) \leqslant R - R_1,$$
$$(M + R)(r + 1)w(n^{-1}(M + R)) < (R - n^{-1}(M + R))^2.$$

Then the sequence $\{v_k\}$ given by Definition 3 remains at each stage in $B_R(0, X)$ and is therefore well-defined over the whole range $1 \leqslant k \leqslant rn$.

Remark. Note that for each r, the two inequalities of Proposition 1 will be satisfied if n is chosen sufficiently large.

Proof of Proposition 1. The sequence $\{v_k\}$ begins with $\| v_0 \| \leqslant R_1$. Let $R_2 = R - n^{-1}(M + R)$. The first inequality of the hypothesis tells us that $R_1 \leqslant R_2$. Hence condition (2) of Definition 2 is valid for all u with $R_2 \leqslant \| u \| \leqslant R$.

Suppose that for a given k, v_{k-1} lies in $B_{R_2}(0, X)$. Since

$$v_k - v_{k-1} = -(n + k)^{-1}v_{k-1} - n^{-1}T(v_{k-1}),$$

and since

$$\| v_{k-1} \| \leqslant R_2 \leqslant R, \qquad \| T(v_{k-1})\| \leqslant M,$$

we know that

$$\| v_k - v_{k-1} \| \leqslant n^{-1}(M + R).$$

and

$$\| v_k \| \leqslant \| v_{k-1} \| + n^{-1}(M + R) \leqslant R_2 + n^{-1}(M + R) \leqslant R.$$

Hence, if v_k for some k lies outside $B_R(0, X)$, the preceding term of the sequence v_{k-1} would have to lie in $B_R(0, X)\backslash B_{R_2}(0, X)$.

On the other hand, for such a value of k, we should have

$$\| v_{k-1} \|^2 \geqslant \| v_k \|^2 + 2(J(v_k), v_{k-1} - v_k)$$

from the fact that the duality mapping J is the subgradient of the convex function $g(x) = \frac{1}{2} \| x \|^2$. We see that

$$(J(v_k), v_k - v_{k-1}) = (J(v_k) - J(v_{k-1}), v_k - v_{k-1}) + (J(v_{k-1}), v_k - v_{k-1}),$$

where

$$\begin{aligned} |(J(v_k) - J(v_{k-1}), v_k - v_{k-1})| &\leqslant \| J(v_k) - J(v_{k-1})\| \cdot \| v_k - v_{k-1} \| \\ &\leqslant w(n^{-1}(M + R))\, n^{-1}(M + R), \end{aligned}$$

and

$$\begin{aligned} (J(v_{k-1}), v_k - v_{k-1}) &= -(n + k)^{-1}(J(v_{k-1}), v_{k-1}) - n^{-1}(J(v_{k-1}), T(v_{k-1})) \\ &\leqslant -(n + k)^{-1} \| v_{k-1} \|^2 \leqslant -n^{-1}(r + 1)^{-1}R_2^2, \end{aligned}$$

since $(J(v_{k-1}), T(v_{k-1})) \geqslant 0$ because v_{k-1} lies in $B_R \backslash B_{R_2}$. Therefore,

$$\| v_k \|^2 \leqslant \| v_{k-1} \|^2 + 2(J(v_k), v_k - v_{k-1}),$$

while

$$(J(v_k), v_k - v_{k-1}) \leqslant -n^{-1}(r + 1)^{-1}R_2^2 + n^{-1}(M + R)w(n^{-1}(M + R)).$$

By the second inequality of the hypothesis and the choice of R_2, the term on the right of this last inequality is negative. Hence

$$\| v_k \| \leqslant \| v_{k-1} \|$$

and v_k cannot lie outside the ball $B_R(0, X)$. Q.E.D.

DEFINITION 4. Under the hypotheses of Proposition 1 we set

$$V(r, n, v_0) = v_{rn}$$

where $\{v_k\}$ is the sequence defined by Definition 3.

THEOREM 1. *Let X be a Banach space with a uniformly convex conjugate space X^*, T an accretive mapping from $B_R(0, X)$ satisfying the conditions for the class* (A) *of Definition* 2. *Let v_0 be an element of $B_{R_1}(0, X)$, and for r and n satisfying the hypotheses of Proposition* 1, *we construct $V_{(r,n,v_0)}$ using Definitions* 3 *and* 4.

Then $V_{(r,n,v_0)}$ converges to the unique solution u_0 of the equation $T(u_0)$ as $r \to \infty$, $n \to \infty$, and

$$(r + 1)w(n^{-1}(M + R)) \to 0,$$

with u_0 constructed as the limit of $V_{(r,n,v_0)}$.

Furthermore, we have the error estimate

$$\| V_{(r,n,v_0)} - u_0 \| \leqslant \{k(r+1)\, w(n^{-1}(M+R)) + k_n n^{-1}\}^{1/2} + q(k_3 r^{-1}(\log r + 1)).$$

for suitable constants k_1, k_2, k_3 which are evaluated below and $r \geqslant k_4$, with k_4 evaluated below.

Remark. When X is a Hilbert space and J is the identity mapping, $w(r) = r$ and the side condition involving r becomes the condition that $r = o(n)$.

We carry through the proof of Theorem 1 by constructing the double sequence of functions $\{u_{r,n}\}$ on the interval $0 \leqslant t \leqslant r$ by setting

$$u_{r,n}(t) = v_{k-1} + \left(t - \frac{k-1}{n}\right) n[v_k - v_{k-1}],$$
$$n^{-1}(k-1) \leqslant t \leqslant n^{-1}k, \qquad 0 \leqslant k \leqslant rn,$$

constructing the solution of the differential equation

$$\frac{du}{dt}(t) + T(u(t)) + (t+1)^{-1}u(t) = 0, \qquad 0 \leqslant t \leqslant r$$

with initial value $u(0) = v_0$ from the approximating functions $u_{r,n}(t)$ for large n, estimating the error in the approximation, and finally obtaining the asymptotic behavior of the solution of the differential equation as $t \to +\infty$.

PROPOSITION 2. *For each fixed r, $u_{r,n}(t)$ converges uniformly to the solution of the equation*

$$\frac{du}{dt}(t) + T(u(t)) + (t+1)^{-1}u(t) = 0,$$

with initial value $u(0) = v_0$. Moreover,

$$\| u_{r,n}(t) - u_{r,m}(t)\| \leqslant 4(M+R)\, rw((M+R)(m^{-1} + n^{-1})) + 4rR^2(m^{-1} + n^{-1}).$$

(This last estimate can be considerably improved in terms of its dependence upon r, but without much consequence to the other results.)

Proof. of Proposition 2. The interval $[0, r]$ can be subdivided into a finite number of intervals on each of which both $u_{r,n}$ and $u_{r,m}$ are linear functions. Consider t in the intersection of the intervals $[n^{-1}(k-1), n^{-1}k]$ and $[m^{-1}(j-1), m^{-1}j]$. Denote the sequence corresponding to the function $u_{r,m}(t)$ by $\{w_j\}$. For t in the intersection of the two intervals, we have:

$$\frac{du_{r,n}(t)}{dt} = -(n+k)^{-1}nv_{k-1} - T(v_{k-1}),$$
$$\frac{du_{r,m}(t)}{dt} = -(m+j)^{-1}mw_{j-1} - T(w_{m-1}).$$

Through their domain of existence, we know that $\|(du_{r,n}/dt)(t)\| \leqslant (M+R)$, $\|(du_{r,m}/dt)(t)\| \leqslant (M+R)$.

Hence, we see that for t as above

$$\| u_{r,n}(t) - v_{k-1} \| \leqslant (M+R)\, n^{-1},$$
$$\| u_{r,m}(t) - w_{j-1} \| \leqslant (M+R)\, m^{-1}.$$

We also know that

$$\begin{aligned}
&\frac{d}{dt}\{\| u_{r,n}(t) - u_{r,m}(t)\|^2\} \\
&\quad = 2\left(J(u_{r,n}(t) - u_{r,m}(t)), \frac{d}{dt} u_{r,n}(t) - \frac{d}{dt} u_{r,m}(t)\right), \\
&\quad = 2(J(u_{r,n}(t) - u_{r,m}(t)), -(n+k)^{-1} n v_{k-1} \\
&\qquad - T(v_{k-1}) + (m+j)^{-1} m w_{j-1} + T(w_{j-1})).
\end{aligned}$$

We may write

$$J(u_{r,n}(t) - u_{r,m}(t)) = J(v_{k-1} - w_{j-1}) + \{J(u_{r,n}(t) - u_{r,m}(t)) - J(v_{k-1} - w_{j-1})\},$$

where the second term has norm at most equal to

$$w(\| u_{r,n}(t) - v_{k-1} \| + \| u_{r,m}(t) - w_{j-1} \|) \leqslant w((M+R)(n^{-1} + m^{-1})).$$

Therefore

$$\begin{aligned}
&\frac{d}{dt}\{\| u_{r,n}(t) - u_{r,m}(t)\|^2\} \\
&\quad \leqslant -2(J(v_{k-1} - w_{j-1}), T(v_{k-1}) - T(w_{j-1})) \\
&\qquad - 2(J(v_{k-1} - w_{j-1}), n(n+k)^{-1} v_{k-1} - m(m+j)^{-1} w_{j-1}) \\
&\qquad + 4(M+R)\, w((M+R)(m^{-1} + n^{-1})) \\
&\quad \leqslant -2(J(v_{k-1} - w_{j-1}), n(n+k)^{-1} v_{k-1} - m(m+j)^{-1} w_{j-1}) \\
&\qquad + 4(M+R)\, w((M+R)(m^{-1} + n^{-1})),
\end{aligned}$$

by the accretivity of T on the ball $B_R(0, X)$.

Since $| t - kn^{-1} | \leqslant n^{-1}$, $| t - jm^{-1} | < m^{-1}$, it follows that

$$| m(m+j)^{-1} - (1+t)^{-1} | < m^{-1},$$
$$| n(n+k)^{-1} - (1+t)^{-1} | < n^{-1}.$$

Thus

$$\| n(n+k)^{-1} v_{k-1} - (1+t)^{-1} v_{k-1} \| \leqslant Rn^{-1},$$
$$\| m(m+j)^{-1} w_{j-1} - (1+t)^{-1} w_{j-1} \| \leqslant Rm^{-1},$$

so that

$$\begin{aligned}
&-2(J(v_{k-1} - w_{j-1}), n(n + k)^{-1}v_{k-1} - m(m + j)^{-1}w_{j-1}) \\
&\quad \leqslant -2(J(v_{k-1} - w_{j-1}), (1 + t)^{-1}(v_{k-1} - w_{j-1}) + 4R^2(n^{-1} + m^{-1}) \\
&\quad \leqslant 4R^2(n^{-1} + m^{-1}).
\end{aligned}$$

Since $\| u_{r,n}(0) - u_{r,m}(0)\| = 0$, we may integrate the inequality

$$\begin{aligned}
&\frac{d}{dt}\{\| u_{r,n}(t) - u_{r,m}(t)\|^2\} \\
&\qquad \leqslant 4R^2(n^{-1} + m^{-1}) + 4(M + R)\, w((M + R)(m^{-1} + n^{-1}))
\end{aligned}$$

step by step on each subinterval, and obtain the desired conclusion.

Using the estimate, we obtain a constructive proof of the convergence of the sequence of functions $u_{r,n}$ for fixed r as $n \to \infty$ to a solution u of the desired initial value proble. Q.E.D.

We now refine the estimate involved in the proof of Proposition 2 by obtaining a sharper estimate for the corresponding error $\| u_{r,n}(t) - u(t)\|$.

PROPOSITION 3. *Let $\{u(t): 0 \leqslant t \leqslant r\}$ be the solution of the initial value problem $u(0) = v_0$ for the differential equation*

$$\frac{du}{dt}(t) + T(u(t)) + (t + 1)^{-1}u(t) = 0.$$

Then for r and n satisfying the hypotheses of Proposition 1, *we have*

$$\| u_{r,n}(t) - u(t)\|^2 \leqslant \tfrac{4}{3}M(r + 1)\, w(n^{-1}(M + R)) + 4R^2(r + 1)\, n^{-1}(M + R), \qquad 0 \leqslant t \leqslant r.$$

In particular,

$$\| V(r, n, v_0) - u(r)\|^2 \leqslant \tfrac{4}{3}M(r + 1)\, w(n^{-1}(M + R)) + 4R^2(r + 1)\, n^{-1}(M + +).$$

Proof of Proposition 3. As in the proof of Proposition 2,

$$\begin{aligned}
&\frac{d}{dt}\{\| u_{r,n}(t) - u(t)\|^2\} \\
&\qquad = -2(J(u_{r,n}(t) - u(t)), T(v_{k-1}) - T(u(t))) - 2(J(u_{r,n}(t) \\
&\qquad\quad - u(t)), (n + k)^{-1}nv_{k-1} - (1 + t)^{-1}u(t)).
\end{aligned}$$

We write

$$J(u_{r,n}(t) - u(t)) = J(v_{k-1} - u(t)) + \{J(u_{r,n}(t) - u(t)) - J(v_{k-1} - u(t))\},$$

in the first term and

$$\begin{aligned}(n+k)^{-1}nv_{k-1} &- (1+t)^{-1}u(t)\\ &= (1+t)^{-1}\{u_{r,n}(t)-u(t)\} + \{(n+k)^{-1}nv_{k-1}-(1+t)^{-1}u_{r,n}(t)\},\end{aligned}$$

in the second term of the equation above. Thus

$$\frac{d}{dt}\{\| u_{r,n}(t)-u(t)\|^2\} \leqslant -2(1+t)^{-1}\| u_{r,n}(t)-u(t)\|^2 + R_n(t),$$

where

$$\begin{aligned}R_n(t) &= 4M \| J(u_{r,n}(t)-u(t)) - J(v_{k-1}-u(t))\|\\ &\quad + 2\| u_{r,n}(t)-u(t)\| \cdot \|(n+k)^{-1}nv_{k-1}-(1+t)^{-1}u_{k,n}(t)\|\\ &\leqslant 4Mw((M+R)n^{-1}) + 4\| u_{r,n}(t)-u(t)\| R(1+t)^{-1}(M+R)n^{-1}\\ &\leqslant 4Mw((M+R)n^{-1}) + 8R^2(1+t)^{-1}(M+R)n^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{d}{dt}\{(1+t)^2\| u_{r,n}(t)-u(t)\|^2\}\\ &\qquad\leqslant 4Mw((M+R)n^{-1})(1+t)^2 + 8R^2(1+t)(M+R)n^{-1},\end{aligned}$$

and

$$\begin{aligned}&(1+t)^2\| u_{r,n}(t)-u(t)\|^2\\ &\qquad\leqslant \frac{(1+t)^3}{4} 4Mw((M+R)n^{-1}) + 4R^2(1+t)^2(M+R)n^{-1}.\end{aligned}$$

Then

$$\begin{aligned}\| u_{r,n}(t)-u(t)\|^2 &\leqslant \frac{4(1+t)}{3} Mw((M+R)n^{-1}) + 4R^2(M+R)n^{-1}\\ &\leqslant \tfrac{4}{3}(r+1)\, Mw((M+R)n^{-1}) + 4R(M+R)n^{-1}.\end{aligned}$$

PROPOSITION 4. *Let $\{u_r(t)\colon 0 \leqslant t \leqslant r\}$ be the solution of the differential equation*

$$\frac{du}{dt}(t) + T(u(t)) + (t+1)^{-1}u(t) = 0, \qquad (0 \leqslant t \leqslant r)$$

with the initial condition $u(0) = v_0$ constructed in the proof of Proposition 2. Any two of these solutions coincide over any interval on which both are defined and hence amalgamate to a single solution $\{u(t)\colon 0 \leqslant t < \infty\}$. For this solution, the first derivative du/dt satisfies the following inequality for its right first upper derivative:

$$\bar{D}^+\left((t+1)\left\|\frac{du}{dt}(t)\right\|\right) \leqslant R(t+1)^{-1}.$$

Proof of Proposition 4. The uniqueness property for solutions follows immediately from their construction as the limit of the sequences $\{u_{r,m}(t)\}$ which coincide for different values of r for common values of m. Hence the solution $\{u(t): t \geqslant 0\}$ is defined on the whole nonnegative real-axis.

Let $h > 0$ be chosen. Set $v(t) = u(t + h)$. Then v satisfies the differential equation:

$$\frac{dv}{dt}(t) + T(v(t)) + (t + h + 1)^{-1}v(t) = 0.$$

Hence

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\| u(t) - v(t)\|^2 \\
&\quad \leqslant \left(J(u(t) - v(t)), \frac{du}{dt}(t) - \frac{dv}{dt}(t)\right) \\
&\quad = -(J(u(t) - v(t)), T(u(t)) - T(v(t)) + (t + 1)^{-1}\{u(t) - v(t)\}) \\
&\qquad + \{(t + h + 1)^{-1} - (t + 1)^{-1}\}(J(u(t) - v(t)), v(t)) \\
&\quad \leqslant -(t + 1)^{-1} \| u(t) - v(t)\|^2 \\
&\qquad + \{(t + 1)^{-1} - (t + h + 1)^{-1}\}R \| u(t) - v(t)\|.
\end{aligned}$$

Hence

$$\frac{d}{dt}\| u(t) - v(t)\| \leqslant -(t + 1)^{-1}\| u(t) - v(t)\| + h(t + 1)^{-2}R.$$

If we now write $v(t)$ as $v_h(t)$ and divide by the positive number h, then for $\delta_h(u) = h^{-1} \| u(t) - u(t + h)\|$, we find that

$$\frac{d}{dt}\delta_h(u)(t) \leqslant -(t + 1)^{-1}\,\delta_h(t) + R(t + 1)^{-2}.$$

We may rewrite this inequality in the form

$$\frac{d}{dt}\{(t + 1)\,\delta_h(u)(t)\} \leqslant R(t + 1)^{-1}.$$

Let $\xi > 0$ be given. Then

$$(t + \xi + 1)\,\delta_h(u)(t + \xi) - (t + 1)\,\delta_h(u)(t) \leqslant R\int_t^{t+\xi} (s + 1)^{-1}\,ds.$$

as $h \to 0+$, $\delta_n u(t) \to \|(du/dt)(t)\|$. Hence

$$(t + \xi + 1)\left\|\frac{du}{dt}(t + \xi)\right\| - (t + 1)\left\|\frac{du}{dt}(t)\right\| \leqslant R\int_t^{t+\xi} (s + 1)^{-1}\,ds.$$

Dividing by $\xi > 0$ and taking the lim sup of the term on the left of the last inequality as $\xi \to 0+$, we obtain the desired inequality. Q.E.D.

PROPOSITION 5. *As $t \to +\infty$, $\| Tu(t)\| \to 0$. More precisely,*

$$\|(Tu)(t)\| \leqslant (t+1)^{-1}\{2R+M\} + R(t+1)^{-1}\log(t+1).$$

Proof of Proposition 5. From Proposition 4, it follows that

$$(t+1)\left\|\frac{du}{dt}(t)\right\| - \left\|\frac{du}{dt}(0)\right\| \leqslant R\int_0^t (s+1)^{-1}\,ds.$$

Note that $\|(Tu)(t)\| \leqslant \|(du/dt)(t)\| + (t+1)^{-1}\| u(t)\|$ and that $\| u(t)\| \leqslant R$ for all $t \geqslant 0$, we obtain the desired inequality. Q.E.D.

PROPOSITION 6. *Let $\delta > 0$ be as given in condition* (4) *for the definition of the class* (A). *Then for*

$$(r+1)^{-1}\{2R + M + R\log(r)\} < \delta/2,$$
$$\| u(t) - u(s)\| \leqslant q(2(r+1)^{-1}(2R + M + R\log(r)))$$

for $s, t \geqslant r$.

Proof of Proposition 6. This is an immediate consequence of Proposition 5 together with the assumption of the uniform continuity property of T^{-1} in the neighborhood of 0 given in condition (4) for the definition of the class (A). Q.E.D.

Proof of Theorem 1. If we combine Propositions 1 through 6, we obtain the complete verification of the conclusion of the Theorem with a precise specification of constants. By Proposition 6, in particular, $u(s)$ converges constructively to a solution u_0 of $T(u_0) = 0$, and the error follows from the uniform errors for $\| u(s) - u(t)\|$. By the preceding propositions, $V(r, n, v_0)$ converges to $u(r)$ as n and r converge appropriately to infinity with the desired control of the error. Q.E.D.

SECTION 2

Let us note that without the assumption that X^* is uniformly convex but with the assumption that T is uniformly continuous on bounded subsets of X, one can carry through a corresponding analysis with an error term based upon the modulus of uniform continuity of T rather than that of J (as is carried through in the corresponding argument in [2]).

It is also shown in [2] that the convergence arguments given in Propositions 5 and 6 for the solution of the equation

$$(du/dt)(t) + T(u(t)) + (t+1)^{-1}u(t) = 0$$

can be carried over to the equation

$$(du/dt)(t) + T(u(t)) + s(t)u(t) = 0$$

provided that s satisfies the following conditions:

(1) The function s is once continuously differentiable, monotone non-increasing and $s(t) \to 0$ as $t \to +\infty$.

(2) The integral

$$\int_0^\infty s(r)\, dr = +\infty.$$

References

1. F. E. Browder, Nonlinear functional analysis and nonlinear partial differential equations, *in* "Proceedings of the 1966 Bratislava Symposium on Differential Equations, Equadiff II. Differential Equations and Their Applications II," 45–64, Bratislava, 1969.
2. F. E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *in* "Nonlinear Functional Analysis," Symposia in Pure Mathematics, Vol. 18, Part 2, Amer. Math. Soc., Providence, R.I., 1976.
3. R. E. Bruck, On the iterative solution of $y \in x + T(x)$ for a bounded monotone operator T in a Hilbert space, *Bull. Amer. Math. Soc.* **79** (1974), 1258–1261.